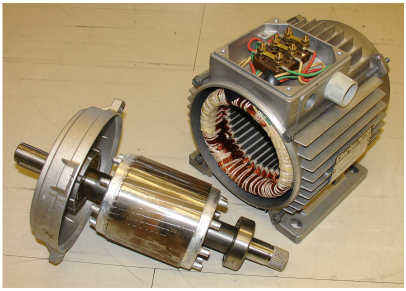
Computing flux linkages in rotational systems

Consider a rotational arrangement with two parts:

- Stator (that remains static)
- Rotor (that rotates)



Source: Wikipedia

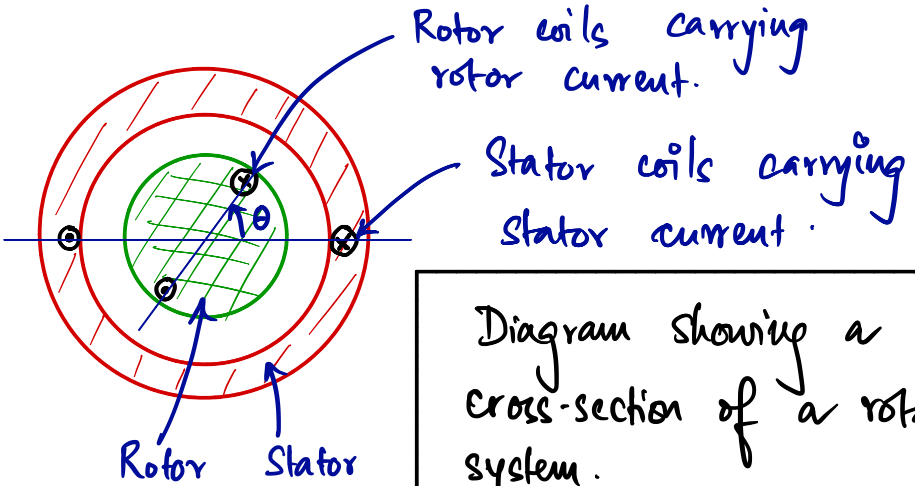
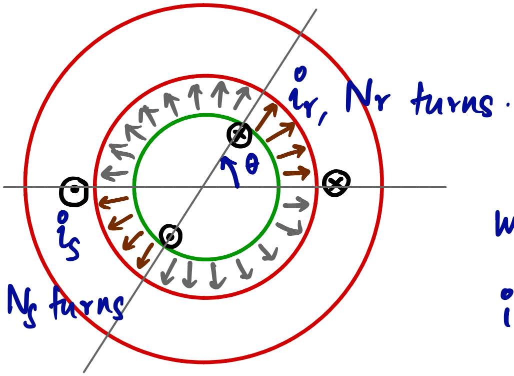


Diagram showing a cross-section of a rotational system.

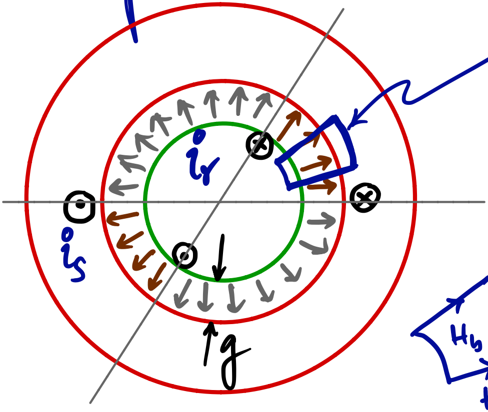
In the following, we will compute the flux linkages λ_s and λ_r with the stator and rotor coils, respectively.



Use symmetry to conclude that $\vec{H}$, magnetic field intensity, is radial in nature!

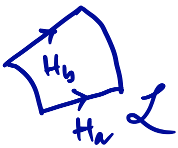
<u>Claim:</u> H between angles 0 to θ is uniform,
between angles θ to π is uniform,
between angles π to $\pi+\theta$ is uniform,
and between angles $\pi+\theta$ to 2π is uniform.

Proof:



Consider this loop $\mathcal{L}$.
Ampere's law states that

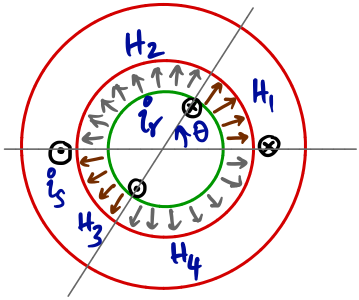
$$\oint_{\mathcal{L}} \vec{H} \cdot d\vec{l} = 0 .$$



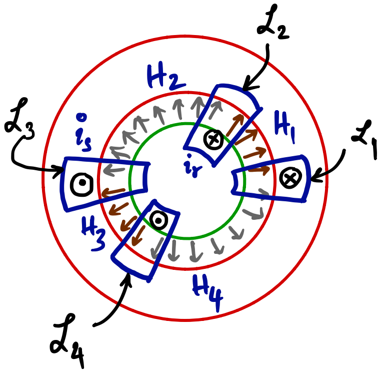
} Expanded view.

$$\Rightarrow (H_a - H_b) \cdot g = 0 \Rightarrow H_a = H_b .$$

$\square$



- Call these uniform radial magnetic field intensities between various angles as H_1, H_2, H_3, H_4, as shown.



- Apply Ampere's law around the loops $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$.

$$(H_4 - H_1) \cdot g = -N_s i_s, \quad : \mathcal{L}_1$$

$$(H_1 - H_2) \cdot g = -N_r i_r, \quad : \mathcal{L}_2$$

$$(H_2 - H_3) \cdot g = N_s i_s, \quad : \mathcal{L}_3$$

$$(H_3 - H_4) \cdot g = N_r i_r. \quad : \mathcal{L}_4$$

- By symmetry, we have $H_1 = -H_3$, and $H_2 = -H_4$. Why? If you reverse the currents, you will interchange the roles of H_1 & H_3, and that of H_2 & H_4.

$$\boxed{\begin{aligned} (H_4 - H_1)\cdot g &= -N_s i_s, \\ (H_1 - H_2)\cdot g &= -N_r i_r, \qquad H_1 = -H_3, \\ (H_2 - H_3)\cdot g &= N_s i_s, \qquad H_2 = -H_4. \\ (H_3 - H_4)\cdot g &= N_r i_r. \end{aligned}}$$

We obtained these from Ampere's law & symmetry.

Let's use these equations to compute $H_1, \dots, H_4$.

$$H_2 - H_3 = N_s i_s / g \Rightarrow H_2 + H_1 = N_s i_s / g.$$

Also, we have $H_1 - H_2 = -N_r i_r / g$

$$\Rightarrow H_1 = \frac{1}{2g}\left(N_s i_s - N_r i_r\right)$$

$$\Rightarrow H_2 = \frac{1}{2g}\left(N_s i_s + N_r i_r\right)$$

- Computing flux linkages λ_s and λ_r.

Denoting magnetic flux densities in the respective locations as B_1, B_2, B_3, B_4, we have $B_i = \mu_0 H_i$, $i = 1, 2, 3, 4$.

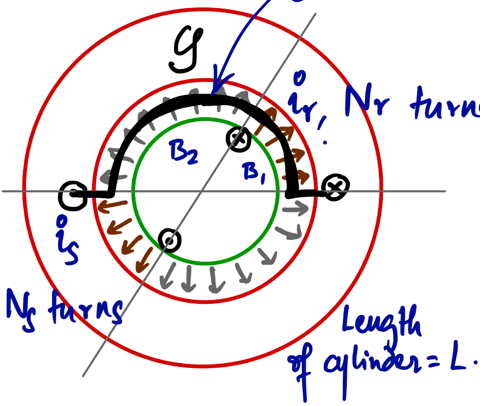
- What is λ_s ?

  Flux linked with the stator coil.

- How do you measure it ?

  Consider an open surface that has the stator coil as its ends.

Consider this en surface, whose cross-section is shown here, and it extends longitudinally along the length of the cylindrical components.



$$\therefore \lambda_s = N_s \int_{\mathcal{G}} \vec{B} \cdot d\vec{s} = N_s \int_{\phi=0}^{\phi=\theta} B_1 . RL\,d\phi + N_s \int_{\phi=\theta}^{\phi=\pi} B_2 . RL\,d\phi,$$

where recall that $B_1 = \mu_0 H_1 = \frac{\mu_0}{2g}(N_s i_s - N_r i_r)$,

and $B_2 = \mu_0 H_2 = \frac{\mu_0}{2g}(N_s i_s + N_r i_r)$.

Let's simplify the expression.

$$\lambda_s = N_s \frac{\mu_0}{2g} R L \left(N_s i_s - N_r i_r \right) \theta + \frac{N_s \cdot \mu_0 R L}{2g} \left(N_s i_s + N_r i_r \right) (\pi - \theta).$$

$$= \frac{\mu_0 R L}{2g} \Big[N_s^2 \cancel{\theta}\, i_s - N_s N_r \theta\, i_r + N_s^2 (\pi - \cancel{\theta})\, i_s + N_s N_r (\pi - \theta)\, i_r \Big]$$

$$= \underbrace{\frac{\mu_0 R L \pi}{2g}}_{:= \mathcal{L}_0} \left[N_s^2 i_s + N_s N_r \left(1 - \frac{2\theta}{\pi} \right) i_r \right].$$

$$= N_s^2 \mathcal{L}_0\, i_s + N_s N_r \mathcal{L}_0 \left(1 - \frac{2\theta}{\pi} \right) i_r.$$

It is of the form

$$\lambda_s = \mathcal{L}_s\, i_s + \mathcal{L}_m(\theta)\, i_r \qquad \begin{cases} \mathcal{L}_s = N_s^2 L_0, \\ \mathcal{L}_m(\theta) = N_s N_r L_0 \left(1 - \frac{2\theta}{\pi} \right). \end{cases}$$

Similarly, one can calculate λ_r.

$$\lambda_r = N_r B_2 \cdot RL(\pi-\theta) + N_r B_3 RL\theta.$$

Use $H_3 = -H_1$.
$\Rightarrow B_3 = -B_1$.

$$= \frac{\mu_0 RL}{2g}\Big[N_r(N_s i_s + N_r i_r)(\pi-\theta) + N_r(N_r i_r - N_s i_s)\theta\Big].$$

$$= \frac{\mu_0 RL}{2g}\Big[N_s N_r(\pi-\theta) i_s + N_r^2(\pi-\cancel{\theta}) i_r + \cancel{N_r^2\theta i_r} - N_s N_r \theta i_s\Big].$$

$$= \underbrace{\frac{\mu_0 RL\pi}{2g}}_{:=\mathcal{L}_0}\Big[N_r^2 i_r + N_s N_r\left(1-\frac{2\theta}{\pi}\right) i_s\Big].$$

$$= N_r^2 \mathcal{L}_0 i_r + N_s N_r \mathcal{L}_0\left(1-\frac{2\theta}{\pi}\right).$$

λ_r is of the form

$$\lambda_r = \mathcal{L}_r i_r + \mathcal{L}_m(\theta) i_s.$$

We explicitly show the dependency on θ.

$$\therefore\quad \lambda_s(\theta) = \mathcal{L}_s\, i_s + \mathcal{L}_m(\theta)\, i_r\,,$$

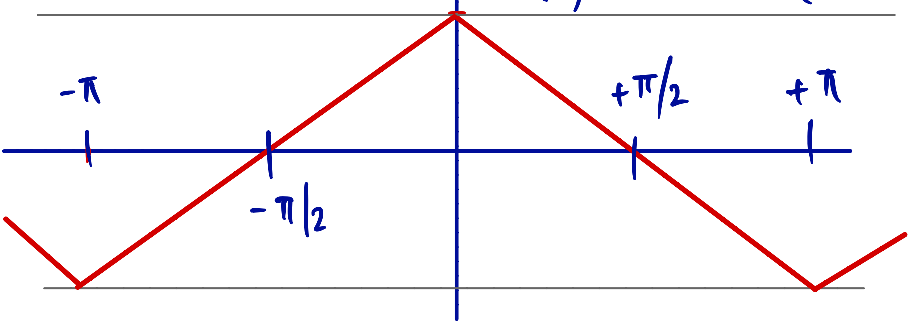
$$\lambda_r(\theta) = \mathcal{L}_m(\theta)\, i_s + \mathcal{L}_r\, i_r\,.$$

$$\Rightarrow \begin{pmatrix} \lambda_s(\theta) \\ \lambda_r(\theta) \end{pmatrix} = \begin{pmatrix} \mathcal{L}_s & \mathcal{L}_m(\theta) \\ \mathcal{L}_m(\theta) & \mathcal{L}_r \end{pmatrix} \begin{pmatrix} i_s \\ i_r \end{pmatrix}$$

$$\Rightarrow \underbrace{\underline{\lambda}(\theta)}_{\text{vector of } \lambda\text{'s}} = \underbrace{\underline{\mathcal{L}}}_{\text{matrix of inductances}}\ \underbrace{\underline{i}}_{\text{vector of } i\text{'s}}\,.$$

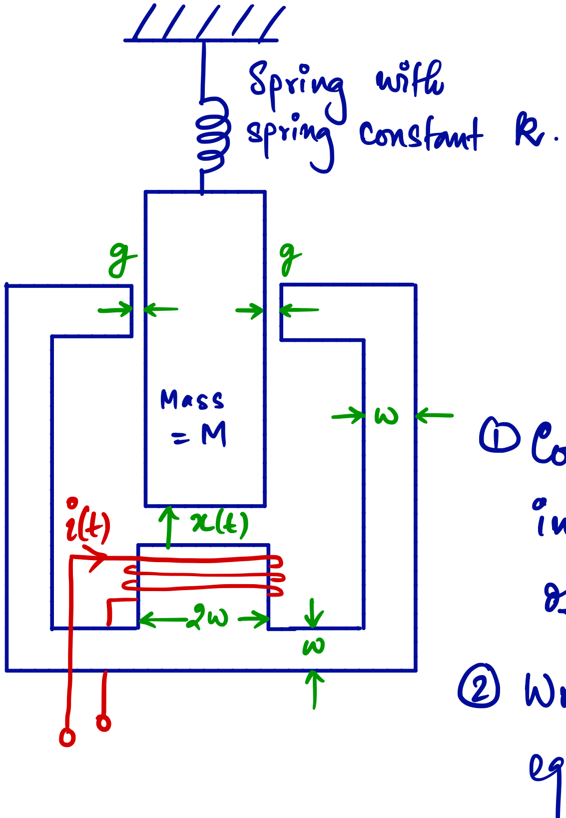
This is an electrically linear system.

$$\mathcal{L}_m(\theta) = \mathcal{L}_0 N_s N_r \left(1 - 2\theta/\pi\right).$$



Dynamical system descriptions of electromechanical systems.

Consider an arrangement shown below.

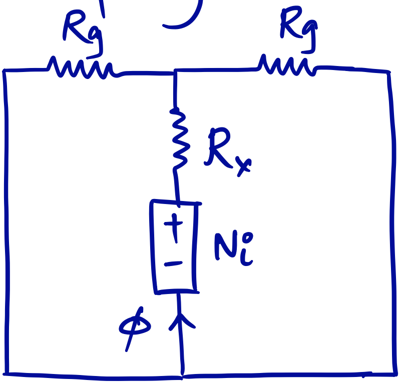


- Uniform depth = w.
- No fringing.
- $x = 0$ corresponds to the spring being uncompressed.

① Compute the voltage induced as a function of time.

② Write the mechanical equation for the block.

① Computing the voltage:



$$\phi = \frac{Ni}{R_x + R_g \parallel R_g} = \frac{Ni}{R_x + \frac{1}{2} R_g},$$

where $R_x = \dfrac{x}{\mu_0 (2w^2)}$, $R_g = \dfrac{g}{\mu_0 w^2}$.

$$\Rightarrow \phi = \frac{2\mu_0 w^2 N i}{x+g} \Longrightarrow \lambda = \frac{2\mu_0 N^2 w^2 i}{x+g}.$$

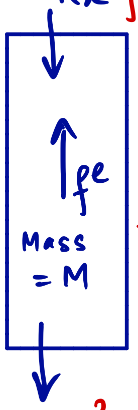
$$\therefore v = \frac{d\lambda}{dt} = (2\mu_0 N^2 w^2)\left[\frac{1}{g+x} \cdot \frac{di}{dt} + i\left(\frac{-1}{(g+x)^2}\right) \cdot \frac{dx}{dt}\right].$$

②

Now, we want to write the mechanical equation for the bar with mass M.

Agenda: Draw free-body diagram.
Write Newton's second law.

Kx } due to the spring getting compressed



} since the bar gets magnetized, it feels this magnetic force. Since the magnetization is the result of electric current, we call it force of electrical origin.

Mg } due to gravity.

Newton's law says:

$$M\ddot{x} = f^e - Mg - Kx.$$

Now, let's calculate f^e. Notice that it would generally be difficult to find the exact magnetization of the bar and compute how it interacts with the magnetic field around. Since this route seems challenging, we compute f^e using energy considerations.

Consider the total energy of the system. Call it W_m. What is the rate at which W_m is changing?

$$\frac{dW_m}{dt} = \underbrace{\begin{pmatrix}\text{power input} \\ \text{to the system}\end{pmatrix}}_{\text{from the current entering the coil.}} - \underbrace{\begin{pmatrix}\text{power output} \\ \text{of the system}\end{pmatrix}}_{\text{from work done by the force of electrical origin.}}$$

$$= i(t) \cdot v(t) - f^e \cdot \frac{dx}{dt}.$$

$$= i \cdot \frac{d\lambda}{dt} - f^e \frac{dx}{dt}.$$

$$\Rightarrow dW_m = i\,d\lambda - f^e\,dx.$$

Let W_m be a function of λ and x. Then Euler's relation is given by

$$dW_m = \frac{\partial W_m}{\partial \lambda} d\lambda + \frac{\partial W_m}{\partial x} \cdot dx.$$

Let's equate these two expressions for W_m.

$$dW_m = i\,d\lambda - f^e\,dx.$$

$$dW_m = \frac{\partial W_m}{\partial \lambda}\,d\lambda + \frac{\partial W_m}{\partial x}\,dx.$$

Equating coefficients, we get

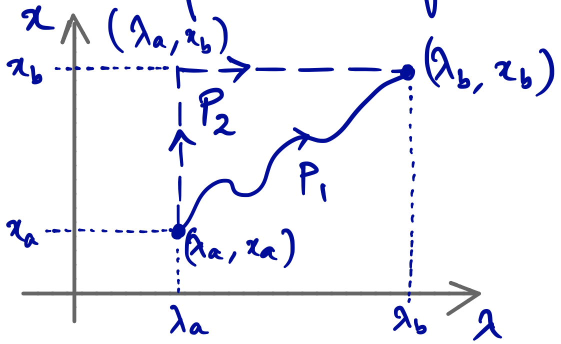
$$i = \frac{\partial W_m}{\partial \lambda}, \quad \& \quad f^e = -\frac{\partial W_m}{\partial x}.$$

Thus to calculate f^e, let's calculate $W_m(\lambda, x)$ and take its partial derivative $-\frac{\partial W_m}{\partial x}$.

- Calculating $W_m(\lambda, x)$.

Suppose we take the system from (λ_a, x_a) to (λ_b, x_b). Notice that if the system is conservative, then the change in W_m is independent of the path taken.

$\therefore$ $W_m(\lambda_b, x_b) - W_m(\lambda_a, x_a)$ is independent of the path. In the adjoining figure, this difference is the same over the paths P_1 & P_2.



$$W_m(\lambda_b, x_b) - W_m(\lambda_a, x_a)$$

$$= \underbrace{\int_{(\lambda_a, x_a)}^{(\lambda_a, x_b)} i\, d\bar{\lambda} - f^e\, d\bar{x}}_{\text{Along this path, } d\bar{\lambda}=0.} + \underbrace{\int_{(\lambda_a, x_b)}^{(\lambda_b, x_b)} i\, d\bar{\lambda} - f^e\, d\bar{x}}_{\text{Along this path, } d\bar{x}=0.}.$$

$$= \int_{(\lambda_a, x_a)}^{(\lambda_a, x_b)} -f^e(\lambda, \bar{x})\, d\bar{x} + \int_{(\lambda_a, x_b)}^{(\lambda_b, x_b)} i(\bar{\lambda}, x)\, d\bar{\lambda}$$

$$\boxed{\begin{aligned} &W_m(\lambda_b, x_b) - W_m(\lambda_a, x_a) \\ &= \underbrace{\int_{(\lambda_a, x_a)}^{(\lambda_a, x_b)} -f^e(\lambda, \bar{x})\, d\bar{x}}_{} + \int_{(\lambda_a, x_b)}^{(\lambda_b, x_b)} i(\bar{\lambda}, x)\, d\bar{\lambda} \end{aligned}}$$

Let's look at this more closely.

$$= \int_{x_a}^{x_b} -f^e(\lambda_a, \bar{x})\, d\bar{x} .$$

Choose $\lambda_a = 0$. Then, there is no force of electrical origin, i.e., $f^e(\lambda_a, \bar{x}) = 0$.

$$\therefore W_m(\lambda_b, x_b) - W_m(0, x_a)$$

$$= \int_{(\lambda_a, x_b)}^{(\lambda_b, x_b)} i(\bar{\lambda}, x_b)\, d\bar{\lambda} .$$

More generally,

$$W_m(\lambda, x) = \overbrace{W_m(0, x)}^{\text{depends on our convention of what we define as a 0-energy level.}} + \int_0^{\lambda} i(\bar{\lambda}, x)\, d\bar{\lambda} .$$

From now on, we will assume $W_m(0, x_n) = 0$.
This is just a convention, and does not affect any calculations.

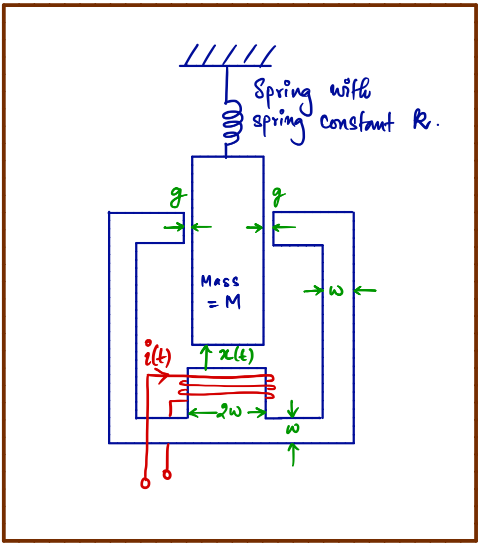
Here onwards, we will calculate

$$\boxed{W_m(\lambda, x) = \int_0^{\lambda} i(\bar{\lambda}, x)\, d\bar{\lambda}.}$$

- Back to calculating f_e.

- Compute $i(\lambda, x)$.
- Invert it to find $\lambda(i, x)$.
- Compute $W_m(\lambda, x) = \int_0^{\lambda} i(\bar{\lambda}, x)\, d\bar{\lambda}$.
- $f^e = -\dfrac{\partial W_m}{\partial x}$.

Then, finish writing Newton's laws with f^e.



• Back to the example.
We already calculated

$$\lambda(i, x) = \frac{2\mu_0 N^2 w^2 i}{x+g}.$$

$$\Rightarrow i(\lambda, x) = \frac{\lambda(x+g)}{2\mu_0 N^2 w^2}.$$

$$\therefore W_m(\lambda, x) = \int_0^{\lambda} i(\bar{\lambda}, x)\, d\bar{\lambda}$$

$$= \int_0^{\lambda} \frac{\bar{\lambda}(x+g)}{2\mu_0 N^2 w^2} \cdot d\bar{\lambda}$$

$$= \frac{\lambda^2 (x+g)}{4\mu_0 N^2 w^2}.$$

$$\therefore f^e = -\frac{\partial W_m}{\partial x} = \frac{-\lambda^2}{4\mu_0 N^2 w^2}$$

Mechanical equation:

$$M\ddot{x} = f^e - Mg - kx$$

$$= \frac{-\lambda^2}{4\mu_0 N^2 w^2} - Mg - kx.$$

Replace λ in terms of i & x.

$$\Rightarrow M\ddot{x} = -\frac{1}{\cancel{4}\cancel{\mu_0} N^2 w^2} \cdot \frac{4\mu_0^{\cancel{2}} N^4 w^4 i^2}{(x+g)^2} - Mg - kx$$

$$= -\frac{\mu_0 N^2 w^2}{(x+g)^2} \cdot i^2 - Mg - kx.$$

- Dynamical system description for the example.
- Electrical equation

$$v = \left(2\mu_0 N^2 w^2\right)\left[\frac{1}{g+x} \cdot \dot{i} + i\left(\frac{-1}{(g+x)^2}\right) \cdot \dot{x}\right].$$

- Mechanical equation

$$M\ddot{x} = -\frac{\mu_0 N^2 w^2}{(x+g)^2} \cdot i^2 - Mg - kx.$$

Co-energy : another route to compute f_e.

Define co-energy for our 1-coil system as

$$W_m' = i\lambda - W_m .$$

Then,
$$\begin{aligned} dW_m' &= i\,d\lambda + \lambda\,di - dW_m \\ &= \cancel{i\,d\lambda} + \lambda\,di - \left[\cancel{i\,d\lambda} - f^e\,dx\right] \\ &= \lambda\,di + f^e\,dx . \end{aligned}$$

If W_m' is a function of i and x, then

$$dW_m' = \frac{\partial W_m'}{\partial i}\,di + \frac{\partial W_m'}{\partial x}\,dx .$$

Equating the coefficient of dx, we get

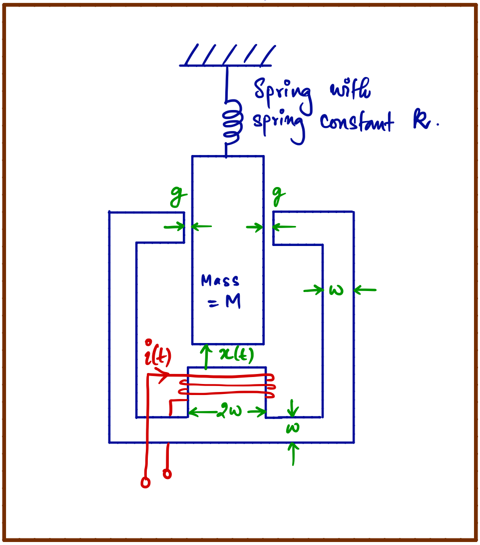
$$\boxed{f^e = \frac{\partial W_m'}{\partial x}} .$$

How do you compute W_m' ?

We will not prove it, but the argument follows very similarly to the derivation of energy W_m. Specifically, we have

$$W_m'(i, x) = \int_0^{i} \lambda(\tilde{i}, x)\, d\tilde{i}.$$

Let's compute f_e through W_m' for our example.



• Back to the example again we already calculated

$$\lambda(i, x) = \frac{2\mu_0 N^2 w^2 i}{x+g}.$$

$$f^e = \frac{-\mu_0 N^2 w^2}{(x+g)^2}\, i^2.$$

$$W_m'(i, x) = \int_0^{i} \frac{2\mu_0 N^2 w^2 \tilde{i}}{x+g} \, d\tilde{i}$$

$$= \frac{\mu_0 N^2 w^2}{(x+g)} i^2 .$$

$$\therefore f^e = \frac{\partial W_m'}{\partial x} = -\frac{\mu_0 N^2 w^2 i^2}{(g+x)^2}$$ ... same expression as calculated from energy W_m.

- Is there any advantage of calculating f^e via W_m' as opposed to W_m?

Ans: You don't need to invert the $\lambda - i$ relationship. This really helps us when we have more than one coil.

Consider an arrangement with two coils carrying currents i_1 and i_2, and there is one geometric variable x.

Then, for an electrically linear system, you will obtain

$$\lambda_1(i_1, i_2, x) = L_1(x)\, i_1 + L_m(x)\, i_2,$$

$$\lambda_2(i_1, i_2, x) = L_m(x)\, i_1 + L_2(x)\, i_2.$$

- Calculating f^e via W_m'.

$$W_m'(i_1, i_2, x) = \int_0^{i_1} \lambda_1(\tilde{i}_1, 0, x)\, d\tilde{i}_1 + \int_0^{i_2} \lambda_2(i_1, \tilde{i}_2, x)\, d\tilde{i}_2.$$

This is NOT zero.
Be careful!

Why? Recall that calculating $W_m'(i_1, i_2, x)$

involves taking a path integral of dW_m' from $(0,0,x)$ to (i_1, i_2, x). In this case, we will take the integral along the path $(0,0,x) \to (i_1, 0, x) \to (i_1, i_2, x)$.

Suppose you have a system with N electrical ports & M mechanical ports.

these refer to current carrying coils.

Let the currents be given by $i_1, i_2, \dots, i_N$, and flux linkages be given by $\lambda_1, \dots, \lambda_N$.

these refer to parts that move within the system.

Let the geometric variables be given by $x_1, \dots x_M$, and the forces due to electrical origin be $f_1^e, \dots, f_M^e$.

Then,

- $$dW_m' = \sum_{j=1}^{N} \lambda_j \, d\dot{i}_j + \sum_{k=1}^{M} f_k^e \, dx_k .$$

- $$W_m'(\dot{i}_1, \dot{i}_2, \dots, \dot{i}_N, x_1, \dots, x_M)$$
$$= \int_0^{\dot{i}_1} \lambda_1(\tilde{\dot{i}}_1, 0, \dots 0, x_1, \dots, x_M) \, d\tilde{\dot{i}}_1$$
$$+ \int_0^{\dot{i}_2} \lambda_2(\dot{i}_1, \tilde{\dot{i}}_2, \dots, 0, x_1, \dots, x_M) \, d\tilde{\dot{i}}_2$$
$$+ \dots + \int_0^{\dot{i}_N} \lambda_N(\dot{i}_1, \dot{i}_2, \dots, \tilde{\dot{i}}_N, x_1, \dots, x_M) \, d\tilde{\dot{i}}_N$$

- $$f_k^e = \frac{\partial W_m'}{\partial x_k} \quad \text{for each } k = 1, \dots, M.$$

Example:

Consider a system with 2 electrical and 2 mechanical ports, whose flux linkages are given by

$$\lambda_1(i_1, i_2, x_1, x_2) = a x_1^2 i_1^3 + b x_2^2 x_1 i_2,$$
$$\lambda_2(i_1, i_2, x_1, x_2) = b x_2^2 x_1 i_1 + c x_2^2 i_2^3,$$

where i_1, i_2 are the currents, and x_1, x_2 describe the geometric variables.

- Is this system electrically linear?
- Calculate f_1^e, f_2^e, the forces due to electrical origin.

Solution: • λ_1 depends on $i_1^3 \Rightarrow$ not electrically linear.

- Computing f_1^e, f_2^e involves two steps:
  - compute W_m'.
  - compute f_1^e, f_2^e.

$$W_m'(i_1, i_2, x_1, x_2)$$

$$= \int_0^{\mathring{i}_1} \lambda_1(\tilde{\mathring{i}}_1, 0, x_1, x_2)\, d\tilde{\mathring{i}}_1$$

$$+ \int_0^{\mathring{i}_2} \lambda_2(\mathring{i}_1, \tilde{\mathring{i}}_2, x_1, x_2)\, d\tilde{\mathring{i}}_2$$

$$= \int_0^{\mathring{i}_1} a x_1^2 \tilde{\mathring{i}}_1^3 \, d\tilde{\mathring{i}}_1 + \int_0^{\mathring{i}_2} \left(b x_2^2 x_1 \mathring{i}_1 + c x_2^2 \tilde{\mathring{i}}_2^3\right) d\tilde{\mathring{i}}_2$$

$$= \frac{1}{4} a x_1^2 \mathring{i}_1^4 + b x_2^2 x_1 \mathring{i}_1 \mathring{i}_2 + \frac{1}{4} c x_2^2 \mathring{i}_2^4$$

$$f_1^e = \frac{\partial W_m'}{\partial x_1} = \frac{1}{2} a \mathring{i}_1^4 x_1 + b x_2^2 \mathring{i}_1 \mathring{i}_2 ,$$

$$f_2^e = \frac{\partial W_m'}{\partial x_2} = 2 b x_2 x_1 \mathring{i}_1 \mathring{i}_2 + \frac{1}{2} c x_2 \mathring{i}_2^4 .$$

For rotational systems, geometric variables are $\theta_1, \ldots, \theta_M$. Instead of forces of electrical origin, you compute torques of electrical origin $T_1^e, \ldots, T_M^e$.